Int. J. Heat Mass Transfer. Vol. 31, No. 5, pp. 1073–1080, 1988
Printed in Great Britain

0017-9310/88 $3.00+0.00
Pergamon Press plc

Non-Fourier propagation of heat pulses in finite medium

J. GEMBAROVIČ and V. MAJERNÍK
Katedra fyziky Pedagogickej fakulty, CS-949 74 Nitra, Czechoslovakia

(*Received* 29 *July* 1987)

Abstract—This paper investigates analytically non-Fourier effects in a finite slab using the hyperbolic heat conduction model. The results for pulsed surface heat flux conditions are compared with those obtained from the standard parabolic heat conduction equation. Detailed analysis of transition between the 'parabolic' and 'hyperbolic' behaviour of the temperature response of the pulse shows, that for $L/\sqrt{(a\tau)} \gtrsim 25$ non-Fourier effects are negligible and the temperature response of the pulse which occurs at $x = 0$, gained in $x = L$, is practically equal to that developed from the parabolic heat conduction equation. Solutions given in this paper have a clear physical interpretation as travelling thermal waves, which enable solutions to this problem for a semi-infinite medium to be written. Series solutions presented here are in convenient form for numerical convergence, and enable one to make a deep analysis of early times of a transient stage in a medium, which plays an important role in the investigation of thermal stresses.

INTRODUCTION

EXPERIMENTS in search of second sound [1–6] clearly demonstrated that for situations involving very short times and temperatures near absolute zero Fourier's law

$$\mathbf{q} = -k \operatorname{grad} T \tag{1}$$

which states that the heat flux $\mathbf{q}$ is proportional to the temperature gradient $\operatorname{grad} T$ (k is the thermal conductivity), become invalid, and the classical parabolic heat conduction equation

$$\frac{\partial T}{\partial t} = a\Delta T \tag{2}$$

cannot be adequately used for the calculation of the desired temperature distribution in a sample after a pulse.

A modified 'non-Fourier' heat flux law, originally proposed by Maxwell [7] (and later by many other investigators [8–13]), is

$$\tau\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -k \operatorname{grad} T \tag{3}$$

where τ is the thermal relaxation time. When equation (3) is incorporated into the continuity equation

$$\operatorname{div} \mathbf{q} = -\rho c\frac{\partial T}{\partial t} \tag{4}$$

where ρ is the mass density and c the specific heat capacity, the hyperbolic heat conduction equation (HHCE) is obtained in the form

$$\tau\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = a\Delta T \tag{5}$$

where $a = k/\rho c$ is the thermal diffusivity.

Analytical solutions of HHCE are available in the literature only for some specific cases

The temperature distribution due to a step change in temperature at the boundary of a semi-infinite medium, was given in refs. [14–16]. Solution of this problem due to a step change of heat flux at the boundary surface is given in ref. [17]. A theoretical prediction of the heat propagation in a semi-infinite medium containing distributed volumetric energy sources is given in ref. [18].

When a region of finite thickness is considered, the analysis of the propagation of a thermal disturbance in a medium becomes an intricate matter, since the released energy travels as a wave while dissipating its energy and reflecting off the boundaries. Taitel [19] presents a solution for a thin layer subject to a step change of temperature on both its sides, while the same problem, but with a step change of temperature on one side, is solved in ref. [20].

Solutions for the temperature and heat flux, as predicted by the HHCE in a region in finite thickness, subjected to a volumetric energy source, are given in ref. [21].

For situations in which analytical solutions are difficult to obtain (e.g. in the case with surface radiation, or temperature-dependent thermal conductivity), HHCE is solved numerically in refs. [22–24].

The existing analytical solutions of HHCE for a finite medium [25] are, however not in convenient form for numerical convergence, and require some lengthy manipulations, particularly for small values of time, which are, from the point of view of the experimenter, the most interesting.

The objective of this investigation is to develop a solution for HHCE in the finite slab exposed to a pulsed surface heat flux, which is rapidly convergent for small values of time. We will discuss in detail the transition between the 'parabolic' and 'hyperbolic' behaviour of the temperature response of the pulse, and we will give the criteria for the onset of non-

NOMENCLATURE

a	thermal diffusivity
c	specific heat capacity of slab
$f(t)$	time distribution of heat flux at $x = 0$
Fo	Fourier number, at/L^2
Fo_1	dimensionless duration of the pulse, at_1/L^2
k	thermal conductivity
L	thickness of slab
p	Laplace variable
$\mathbf{q}$	heat flux vector
Q	amount of energy of a pulse per unit area
t	time
t'	integral variable
t^*	time needed for appearance of the front of temperature disturbance in the plane $x = L$
t_1	duration of the heat pulse
V	dimensionless temperature
Ve	Vernotte number
x	spatial variable
X	dimensionless spatial variable, x/L
$T(x, t)$	temperature in space–time point x, t.

Greek symbols

Δ	difference (Laplace operator)
ρ	mass density
τ	thermal relaxation time.

Subscripts

b	back side
f	front side.

Fourier effects in a medium. Analytical solutions given in this paper explain a structure of the temperature response on a pulse experimentally gained by Ackerman *et al.* [2].

ANALYSIS

Consider a slab of thickness L, initially at the equilibrium temperature $T(x, 0) = 0$, with constant thermal properties and insulated boundaries. At time $t = 0$ the external surface at $x = 0$ is suddenly exposed to a time-dependent heat flux with prescribed rate $f(t)$ per unit time.

Instantaneous pulse

First, we develop a solution of one-dimensional HHCE

$$\tau\frac{\partial^2 T}{\partial t^2}+\frac{\partial T}{\partial t}=a\frac{\partial^2 T}{\partial x^2},\quad 0 \leqslant x \leqslant L, t \geqslant 0 \tag{6}$$

subject to the initial conditions

$$T(x, 0) = 0 \tag{7}$$

$$\frac{\partial T}{\partial t}(x, 0) = 0 \tag{8}$$

$$q(x, 0) = 0 \tag{9}$$

and the boundary conditions

$$q(0, t) = -k\frac{\partial T}{\partial x}(0, t)-\tau\frac{\partial q}{\partial t}(0, t) = Q\delta(t) \tag{10}$$

$$\frac{\partial T}{\partial x}(L, t) = 0 \quad \text{or} \quad q(L, t) = 0. \tag{11}$$

Here Q is a constant and $\delta(t)$ is the Dirac delta function. Conditions (10) and (11) represent the impulse surface heat source and insulated boundary, respectively.

In our previous paper [26] we presented a solution of HHCE (6) subjected to initial conditions (7)–(9) and boundary conditions

$$k\left.\frac{\partial T}{\partial x}\right|_{x=0} = -Q\delta(t) \tag{12}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=L} = 0 \tag{13}$$

which represent the insulated surface at $x = L$, and the prescribed heat flux at $x = 0$

$$q(0, t) = \frac{Q}{\tau}e^{-t/\tau}u(t) \tag{14}$$

where $u(t)$ is the Heaviside unit step function. Expression (14) follows from the one-dimensional equation (3), in which the right-hand side is given by equation (12).

If we apply the Laplace transformation to equation (6) by taking into account initial conditions (7)–(9), then we obtain the following subsidiary equation:

$$a\frac{d^2\bar{T}}{dx^2}-p(p\tau+1)\bar{T} = 0,\quad 0 \leqslant x \leqslant L \tag{15}$$

with conditions

$$\frac{d\bar{T}}{dx} = -\frac{Q}{k}(1+p\tau),\quad x = 0 \tag{16}$$

$$\frac{d\bar{T}}{dx} = 0,\quad x = L \tag{17}$$

where

$$\bar{T} = \int_0^\infty T(x, t)\,e^{-pt}\,dt$$

is the Laplace transform of temperature $T(x, t)$.

The solution of equation (15) with respect to equation (16), and equation (17) is

$$\bar{T}(x,p) = \frac{Q}{kh}\frac{(1+p\tau)}{(1-\mathrm{e}^{-2hL})}[\mathrm{e}^{-hx}+\mathrm{e}^{-h(2L-x)}] \quad (18)$$

where

$$h = \sqrt{(\tau p(p+1/\tau)/a)}.$$

Since we seek a solution of equation (6) for short times, we expand the term $(1-\mathrm{e}^{-2hL})^{-1}$ into a series for a large h. Here the right-hand side of equation (18) has the form

$$\bar{T}(x,p) = \frac{Q}{k}(1+p\tau)\sum_{m=0}^{\infty}\frac{1}{h} \times \{\mathrm{e}^{-h(2mL+x)}+\mathrm{e}^{-h(2mL+2L-x)}\}. \quad (19)$$

Using a table of transforms [27] we can find the inverse transformation of equation (19)

$$T(x,t) = H(x,t)+\tau\frac{\partial H(x,t)}{\partial t} \quad (20)$$

where

$$H(x,t) = \frac{Q}{k}\sqrt{\left(\frac{a}{\tau}\right)}\exp\left(-\frac{t}{2\tau}\right) \times \sum_{m=0}^{\infty}\left\{I_0\left(\frac{1}{2\tau}\sqrt{\left(t^2-(2mL+x)^2\frac{\tau}{a}\right)}\right) \times u\left[t-\sqrt{\left(\frac{\tau}{a}\right)}(2mL+x)\right] + I_0\left(\frac{1}{2\tau}\sqrt{\left(t^2-(2mL+2L-x)^2\frac{\tau}{a}\right)}\right) \times u\left[t-\sqrt{\left(\frac{\tau}{a}\right)}(2mL+2L-x)\right]\right\} \quad (21)$$

and $I_0(z)$ is the modified Bessel function of the first kind of order zero.

The first term in equation (21) describes the flow of thermal waves from the plane $x = 0$ to the right, and the second one describes the waves reflected from the plane $x = L$ moving to the left.

Extended pulse

Now suppose that the instantaneous sources occur at $t = t'$, i.e. we shift the time origin to $-t'$ by writing $t-t'$ in place of t. Suppose also that there is a sequence of instantaneous sources $Q/\rho c$ in the interval 0 to t_1. Let this time sequence be of magnitude proportional to $f(t')$, where $f(t')$ is a non-dimensional function. To obtain the temperature distribution at time t, we integrate from 0 to t when $t < t_1$ or from 0 to t_1 when $t > t_1$. In the latter case we have

$$T(x,t) = \int_0^{t_1} f(t')\left[H(x,t-t')-\tau\frac{\partial H(x,t-t')}{\partial t'}\right]\mathrm{d}t'. \quad (22)$$

Semi-infinite medium

From the foregoing simple physical meaning of the individual terms of equation (21), we can easily determine the solution of HHCE (6) subject to conditions (7)–(10) for the extended pulses in a semi-infinite medium $x \geqslant 0$. For this purpose it is sufficient to omit all the terms which represent the waves reflected from both boundary planes, and to take only the first term with $m = 0$, which represents the wave moving from the plane $x = 0$ to the right. Here function H takes the form

$$H(x,t) = \frac{Q}{k}\sqrt{\left(\frac{a}{\tau}\right)}\exp\left(-\frac{t}{2\tau}\right) \times I_0\left[\frac{1}{2\tau}\sqrt{\left(t^2-x^2\frac{\tau}{a}\right)}\right]u\left[t-x\sqrt{\left(\frac{\tau}{a}\right)}\right].$$

Non-dimensional formulae

If we want to express the temperature in a medium as a fraction of the steady-state temperature after pulse, i.e.

$$\frac{Q}{\rho c L}\int_0^{t_1} f(t')\,\mathrm{d}t'$$

then equation (22) has the form

$$V(x,t) = \frac{\int_0^{t_1} f(t')\left[H(x,t-t')-\tau\frac{\partial H(x,t-t')}{\partial t'}\right]\mathrm{d}t'}{\int_0^{t_1} f(t')\,\mathrm{d}t'} \quad (23)$$

and the dimensionless form of $H(x, t)$ is

$$H(x,t) = \frac{L}{\sqrt{(a\tau)}}\exp\left(-\frac{t}{2\tau}\right) \times \sum_{m=0}^{\infty}\left\{I_0\left[\frac{1}{2\tau}\sqrt{\left(t^2-(2mL+x)^2\frac{\tau}{a}\right)}\right] \times u\left[t-(2mL+x)\sqrt{\left(\frac{\tau}{a}\right)}\right] + I_0\left[\frac{1}{2\tau}\sqrt{\left(t^2-(2mL+2L-x)^2\frac{\tau}{a}\right)}\right] \times u\left[t-(2mL+2L-x)\sqrt{\left(\frac{\tau}{a}\right)}\right]\right\}. \quad (24)$$

For convenience in the subsequent numerical analysis, the following dimensionless quantities are introduced:

$$Fo = \frac{at}{L^2} \quad \text{(Fourier number)} \quad (25)$$

$$Ve = \frac{\sqrt{(a\tau)}}{L} \quad \text{(Vernotte number)} \quad (26)$$

$$X = \frac{x}{L} \quad \text{(dimensionless coordinate).} \qquad (27)$$

The function H, given by equation (24), after substituting equations (25)–(27) takes the form

$$H(X, Fo) = \frac{1}{Ve}\exp\left(-\frac{Fo}{2Ve^2}\right) \times \sum_{m=0}^{\infty}\left\{I_0\left(\frac{1}{2Ve^2}\sqrt{(Fo^2-(2m+X)^2Ve^2)}\right) \times u[Fo-(2m+X)Ve] + I_0\left(\frac{1}{2Ve^2}\sqrt{(Fo^2-(2m+2-X)^2Ve^2)}\right) \times u[Fo-(2m+2-X)Ve]\right\}. \qquad (28)$$

The dependence of temperature rise on time (Fourier number) in the plane $X = 1$ is of great interest from the experimental point of view. From equations (23) and (24) we immediately obtain

$$V(1, Fo) = \frac{\int_0^{Fo_1} f(Fo')\left[H(1, Fo-Fo') - Ve^2\frac{\partial H(1,\ Fo-Fo')}{\partial Fo'}\right]\mathrm{d}Fo'}{\int_0^{Fo_1} f(Fo')\,\mathrm{d}Fo'} \quad \text{for} \quad Fo > Fo_1 \qquad (29)$$

where

$$Fo_1 = \frac{at_1}{L^2}, \quad Fo' = \frac{at'}{L^2}. \qquad (30)$$

Relation (29) represents the formula for the calculation of the time course of the temperature rise at $x = L$ for the finite medium limited by the thermally insulated planes $x = 0$ and L, where the heat flux with the prescribed rate $f(t)$, $(t \in \langle 0, t_1\rangle)$, occurs at the plane $x = 0$.

In the special case of a rectangular pulse with duration Fo_1, when

$$f(Fo) = u(Fo) - u(Fo - Fo_1) \qquad (31)$$

we obtain from equation (29) for $Fo > Fo_1$ the expression

$$V(1, Fo) = \frac{2Ve}{Fo_1}\sum_{m=0}^{\infty}\left\{\frac{1}{Ve^2} \times \int_0^{Fo_1}\exp\left(-\frac{Fo-Fo'}{2Ve^2}\right)I_0\left(\frac{1}{2Ve^2} \times \sqrt{[(Fo-Fo')^2-(2m+1)^2Ve^2]}\right)\mathrm{d}Fo' \times u[Fo-(2m+1)Ve] + \exp\left(-\frac{Fo}{2Ve^2}\right) \times I_0\left(\frac{1}{2Ve^2}\sqrt{(Fo^2-(2m+1)^2Ve^2)}\right) \times u[Fo-(2m+1)Ve] - \exp\left(-\frac{Fo-Fo_1}{2Ve^2}\right) \times I_0\left(\frac{1}{2Ve^2}\sqrt{[(Fo-Fo_1)^2-(2m+1)^2Ve^2]}\right) \times u[Fo-Fo_1-(2m+1)Ve]\right\}. \qquad (32)$$

If $Fo \leqslant Fo_1$, then in equation (23) Fo occurs instead of Fo_1, and the last term in equation (32) becomes equal to zero.

RESULTS AND DISCUSSIONS

Numerical results displaying the development of the temperature distribution arising from a pulsed surface heat source at $x = 0$ on a slab with thickness L are now presented.

Figure 1 shows the temperature distribution vs position after a rectangle pulse with duration $Fo_1 = 0.01$ at various times Fo, and for $1/Ve = 2$. Here we note that the series solution, utilized to compute the temperature distributions

$$V(X, Fo) = \frac{Ve^2}{Fo_1}\left\{\frac{1}{Ve}\int_0^{Fo_1} H(X, Fo-Fo')\,\mathrm{d}Fo' + [H(X, Fo) - H(X, Fo-Fo_1)]\right\} \qquad (33)$$

where $H(X, Fo)$ is given by equation (28), is in very convenient form for the numerical convergence for a

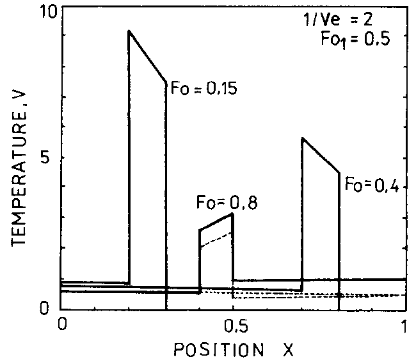

FIG. 1. Temperature distribution vs position after a rectangle pulse.

small value of time—that is in contrast to the solution of HHCE for the similar problems obtained by the method of Fourier transformation [21, 25].

The dominant feature in this figure is that a surface flux of heat gives rise to a thermal wave which travels in the medium at a finite velocity, given by $\sqrt{(a/\tau)}$. The width of this wave ΔX depends on the pulse duration Fo_1, and the speed of propagation of heat disturbance $\sim 1/Ve$, namely

$$\Delta X = \frac{Fo_1}{Ve}. \quad (34)$$

The height of the discontinuity on the front side of the wave ΔV_f decays exponentially with time Fo, and depends on both Fo_1 and Ve. It can be determined from

$$\Delta V_f = \frac{Ve}{Fo_1}\exp\left(-\frac{Fo}{2Ve^2}\right) \quad (35)$$

which can be easily evaluated from equation (33). Similarly, we can find the peak of discontinuity on the back side of the wave

$$\Delta V_b = \frac{Ve}{Fo_1}\exp\left(-\frac{Fo-Fo_1}{2Ve^2}\right). \quad (36)$$

The wave front is dissipating its energy along its path, and the time needed for its decay can be estimated from equation (36) or equation (35).

The temperature discontinuities at the front and back side of the wave are in this case determined by the existence of discontinuities in the heat flux distribution function on the surface as defined by condition (31). The existence of a thermal wave front is conditioned by the fact, that the expression in square brackets in equation (33) is non-zero. This expression represents the wave part of the solution. The integral in equation (33) represents the dissipative part of the solution.

The curve for $Fo = 0.8$ (Fig. 1) represents the distribution of temperature vs position in the slab after the wave front has been reflected from the insulated boundary surface at $X = 1$, and returned travelling in the negative X-direction. A detailed discussion of this process of reflection of the wave front at both surfaces was given in ref. [21].

The effect of Vernotte number on temperature vs time plot at $X = 1$ is demonstrated in Fig. 2. Here we are observing the temperature responses on a rectangle pulse on $X = 0$, with the duration $Fo_1 = 0.05$. As we see from this analysis, curves for $1/Ve \gtrsim 25$ are practically identical with the parabolic one (for $1/Ve \to \infty$), given by

$$V(1, Fo \leqslant Fo_1) = \frac{2}{\sqrt{\pi}\,Fo_1}\sum_{n=0}^{\infty}\int_0^{Fo}\frac{1}{\sqrt{(Fo-Fo')}} \times \exp\left[-\frac{(2n+1)^2}{4(Fo-Fo')}\right]\mathrm{d}Fo'$$

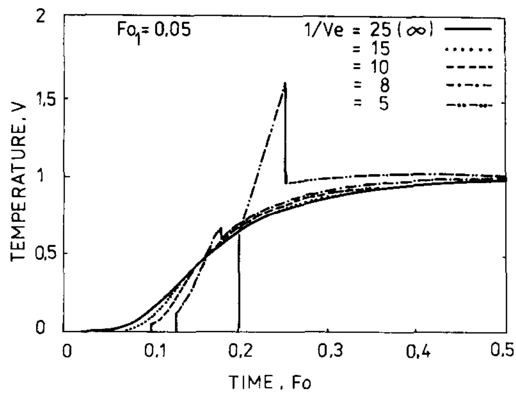

FIG. 2. Effect of Vernotte number on temperature vs time plot at $X = 1$.

$$V(1, Fo > Fo_1) = \frac{2}{\sqrt{\pi}\,Fo_1}\sum_{n=0}^{\infty}\int_0^{Fo_1}\frac{1}{\sqrt{(Fo-Fo')}} \times \exp\left[-\frac{(2n+1)^2}{4(Fo-Fo')}\right]\mathrm{d}Fo' . \quad (37)$$

The inequality $1/Ve \gtrsim 25$ represents the condition of the occurrence of a difference in the temperature responses given by the solution of HHCE, namely relation (32), and the solution of the parabolic heat conduction equation given by equations (37). From the definition of Vernotte number (26) follows, that this condition can also be written in the form

$$25\tau \gtrsim t^* \quad (38)$$

where t^* is the time needed for appearance of the front of the thermal wave in the plane $X = 1$.

The common value of Ve at room temperature and for the typical solids ($L \sim 10^{-2}$ m, $\tau \sim 10^{-11}$ s, $a \sim 10^{-5}$ m^2 s^{-1}) is of the order of 10^{-6}. That means that the effect of the finite velocity of propagation of the thermal disturbance in the common materials with typical dimensions of the samples does not appear. However, at the low temperatures the values of a and τ increase, therefore at a certain value of L the conditions can become favourable for the observation of the temperature responses at the pulses which are supposed by the solution of HHCE.

From the point of view of the analysis of propagation of the thermal wave, presented when discussing Fig. 1, it means that if $1/Ve \gtrsim 25$, then the thermal wave front is damped so, that when it arrives at $X = 1$, it is negligibly small. The satisfaction of condition (38) does not represent a guarantee that the temperature inside the slab is given by the solution of the parabolic heat conduction equation. For the values $1/Ve \approx 25$ and for $Fo < Ve$ it is certainly not the case, especially for the short pulses $Fo_1 \ll Fo$. Inside the slab the temperature distribution vs position can be considerably different from that, which is described by the solution of the classical equation of heat conduction of the parabolic type.

Similarly as in Fig. 1 we can find the formulas for

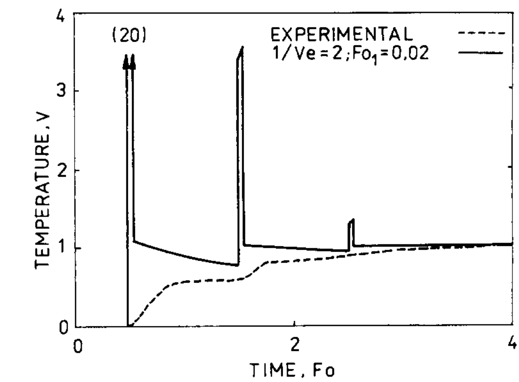

FIG. 3. Comparison of the theoretically predicted temperature response on the pulse and experimentally gained curve.

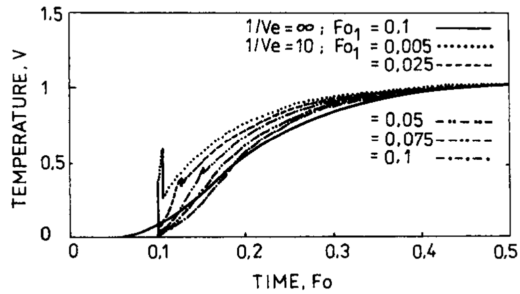

FIG. 4. Effect of pulse duration on temperature vs time plot at $X = 1$.

the determination of the discontinuities in temperature registered in the plane $X = 1$. The first appearance of the front of the wave in this plane occurs at the time $Fo = Ve$. For this discontinuity it follows from equation (35) that

$$\Delta V = \frac{Ve}{Fo_1}\exp\left(-\frac{1}{2Ve}\right). \tag{39}$$

From equation (32) we can conclude that the second discontinuity, which represents the first appearance of the back side of the thermal wave front in the plane $X = 1$, is given also by equation (39).

The heights of the next discontinuities (nth) are given by

$$\Delta V = \frac{Ve}{Fo_1}\exp\left(-\frac{2n-1}{2Ve}\right), \quad n = 1,2,3,\ldots \tag{40}$$

From formula (40) the phenomenological criterion for the appearance of the second 'jump' in the temperature vs time plot can be determined. This is commonly regarded as a second sound echo [2]. For example, for $Fo_1 = 10^{-3}$ and $n = 2$, we have from formula (40), that $\Delta V > 10^{-2}$ if $1/Ve < 6.4$.

The conditions on the existence of second sound in dielectric solids, taken from the point of view of the phonon transport mechanism, are developed in ref. [29].

A comparison of our theoretically predicted temperature response and those obtained experimentally by Ackerman *et al.* [2] is presented in Fig. 3. Here we take the value $Fo_1 = 0.02$, and for $1/Ve$ we choose the value equal to 2, which are in the agreement with the data reported in ref. [2]. Probably due to the finite response time of the detector used, the experimental curve obtained for solid helium differs from that, calculated here. Nevertheless, as can be seen from Fig. 3, essential features of the experimental curve are satisfactorily explained by our hyperbolic solution presented here.

The effect of the pulse duration Fo_1 is demonstrated in Fig. 4, where the temperature field vs time at the boundary plane $X = 1$ is shown. As follows from formula (40), increasing the pulse duration Fo_1 decreasing the energy concentration in the peak, the height of the wave front becomes smaller. We observe from Fig. 4, that for $Fo_1 \geqslant 0.075$ the hyperbolic curves loose their discontinuous character (for a given value $1/Ve$), and they resemble more and more the curves determined by the solution of the classical heat conduction equation.

If we use the possibility of the analytical description of the integral occurring in equation (33) for $X = 0$, whereby we use the formula presented in ref. [28]

$$\int_0^z e^{-t} I_0(t)\,dt = z\,e^{-z}[I_0(z)+I_1(z)] \tag{41}$$

then the temperature on the front side of the slab, for early stages of the transient (for $Fo < 2Ve$), can be expressed in the form

$$V(0,Fo) = \frac{Ve}{Fo_1}\exp\left(-\frac{Fo}{2Ve^2}\right)\left\{\left[\left(1+\frac{Fo}{Ve^2}\right)\times I_0\left(\frac{Fo}{2Ve^2}\right)+\frac{Fo}{Ve^2}I_1\left(\frac{Fo}{2Ve^2}\right)\right]u(Fo) - \exp\left(-\frac{Fo_1}{2Ve^2}\right)I_0\left(\frac{Fo-Fo_1}{2Ve^2}\right)u(Fo-Fo_1)\right\}. \tag{42}$$

This relation describes the temperature of the surface of a slab, suddenly exposed to an intensive laser pulse of short duration, which is a problem of current interest [30].

Unlike the relation for the surface temperature of the slab, derived from the parabolic solution, which starts at time instant $Fo = 0$ from zero and is represented by a continuous function

$$V(0,Fo \leqslant Fo_1) = \frac{1}{\sqrt{\pi}Fo_1} \times \sum_{n=1}^{\infty}\int_0^{Fo}\left[1+2\exp\left(-\frac{n^2}{Fo-Fo'}\right)\right]\frac{dFo'}{\sqrt{(Fo-Fo')}}$$

$$V(0,Fo > Fo_1) = \frac{1}{\sqrt{\pi}Fo_1} \times \sum_{n=1}^{\infty}\int_0^{Fo_1}\left[1+2\exp\left(-\frac{n^2}{Fo-Fo'}\right)\right]\frac{dFo'}{\sqrt{(Fo-Fo')}} \tag{43}$$

our solution (42) starts from the value Ve/Fo_1, and that value has also the second discontinuous jump in time Fo_1 (when $Fo_1 < 2Ve$), toward the lower temperature. In time $Fo = 2Ve$ the reflected thermal wave front arrives at the front of the slab, and it is necessary to add to equation (42) a further term which contains the integral.

This above-mentioned double temperature jump can, in the vicinity of the exposed surface, give rise to a considerable thermomechanical tension, and may lead to the destruction of this surface. All the next possible discontinuous jumps in temperature are smaller in magnitude than the first two, therefore they are, from the point of view of material destruction, not important.

CONCLUSIONS

The Laplace transform method has been applied to develop a solution of one-dimensional HHCE in an insulated finite slab with a surface heat flux boundary condition. The numerical analysis of given solutions shows that in the slab moves a dumping thermal wave which travels through the medium at a constant finite speed, and is reflected by both boundary surfaces, while dissipating its energy along its path.

We present the relations which describe the parameters of this wave and its spatial and temporal development for the rectangular pulsed surface heat source. From the analysis of the transition area between the hyperbolic and parabolic description of the temperature responses on the pulses, determined at $X = 1$, it follows the condition for the onset of non-Fourier effects, $1/Ve \lesssim 25$, in the said arrangement of the heat source, and the temperature detector. The gained solutions are suitable from the point of view of numerical convergence in early stages of the transient. As an illustration of the said topic we present an example for the calculation of the dependence of temperature vs time in the front of a slab. From the point of view of the occurrence of extremal thermotension in the material, the mentioned time interval becomes critical.

The simple physical interpretation of the individual terms in given solutions of HHCE makes it possible to find the solutions to HHCE, with the same boundary and initial conditions also for a semi-infinite medium.

REFERENCES

1. V. Peshkov, 'Second sound' in helium II, *J. Phys. U.S.S.R.* **VIII,** 381 (1944).
2. C. C. Ackerman, B. Bertman, H. A. Fairbank and R. A. Guyer, Second sound in solid helium, *Phys. Rev. Lett.* **16,** 789–790 (1966).
3. B. Bertman and D. J. Sandiford, 'Second sound' in solid helium, *Scient. Am.* **222,** 92–101 (1970).
4. H. E. Jackson and C. T. Walker, Thermal conductivity, second sound, and phonon–phonon interactions in NaF, *Phys. Rev.* **B3,** 1428–1439 (1971).
5. S. J. Rogers, Transport of heat and approach to second sound in some isotopically pure alkali-halide crystals, *Phys. Rev.* **B3,** 1440–1457 (1971).
6. M. Greenstein, M. A. Tamor and J. P. Wolfe, Propagation of laser generated heat pulses in crystals at low temperature: spatial filtering of ballistic phonons, *Phys. Rev.* **B26,** 5604–5610 (1982).
7. J. C. Maxwell, On the dynamical theory of gases, *Phil. Trans. R. Soc.* **157,** 49–88 (1867).
8. P. Vernotte, Les paradoxes de la théorie continue de l'equation de la chaleur, *C.R.* **246,** 3154–3155 (1958).
9. C. Cattaneo, A form of heat conduction equation which eliminates the paradox of instantaneous propagation, *C.R.* **247,** 431 (1958).
10. M. E. Gurtin and A. C. Pipkin, A general theory of heat conduction with finite wave speeds, *Archs Ration. Mech. Analysis* **31,** 113–126 (1968).
11. M. Chester, Second sound in solids, *Phys. Rev.* **131,** 2013–2015 (1963).
12. A. V. Luikov, Application of irreversible thermodynamics, *Int. J. Heat Mass Transfer* **9,** 139–152 (1966).
13. H. D. Weymann, Finite speed of propagation in heat conduction, diffusion, and viscous shear motion, *Am. J. Phys.* **35**(6), 488–496 (1967).
14. B. B. Mikic, A model rate equation for transient thermal conduction, *Int. J. Heat Mass Transfer* **10,** 1899–1904 (1967).
15. K. J. Baumeister and T. D. Hamill, Hyperbolic heat-conduction equation—a solution for the semi-infinite body problem, *J. Heat Transfer* **91,** 543–548 (1969).
16. K. J. Baumeister and T. D. Hamill, Hyperbolic heat-conduction equation—a solution for the semi-infinite body problem, *J. Heat Transfer* **93,** 126–127 (1971).
17. M. J. Maurer and H. A. Thompson, Non-Fourier effects at high heat flux, *J. Heat Transfer* **95,** 284–286 (1973).
18. B. Vick and M. N. Ozişik, Growth and decay of a thermal pulse predicted by the hyperbolic heat conduction equation, *J. Heat Transfer* **105,** 902–907 (1983).
19. Y. Taitel, On the parabolic, hyperbolic and discrete formulation of the heat conduction equation, *Int. J. Heat Mass Transfer* **15,** 369–371 (1972).
20. G. F. Carey and M. Tsai, Hyperbolic heat transfer with reflection, *Numer. Heat Transfer* **5,** 309–327 (1982).
21. M. N. Ozişik and B. Vick, Propagation and reflection of thermal waves in a finite medium, *Int. J. Heat Mass Transfer* **27,** 1845–1854 (1984).
22. D. E. Glass, M. N. Ozişik, D. S. McRae and B. Vick, On the numerical solution of hyperbolic heat conduction, *Numer. Heat Transfer* **8,** 497–504 (1985).
23. D. E. Glass, M. N. Ozişik and B. Vick, Hyperbolic heat conduction with surface radiation, *Int. J. Heat Mass Transfer* **28,** 1823–1830 (1985).
24. D. E. Glass, M. N. Ozişik, D. S. McRae and B. Vick, Hyperbolic heat conduction with temperature-dependent thermal conductivity, *J. Appl. Phys.* **59,** 1861–1865 (1986).
25. J. I. Frankel, B. Vick and M. N. Ozişik, Flux formulation of hyperbolic heat conduction, *J. Appl. Phys.* **58,** 3340–3345 (1985).
26. J. Gembarovič and V. Majerník, Determination of thermal parameters of relaxation materials, *Int. J. Heat Mass Transfer* **30,** 199–201 (1987).
27. V. A. Ditkin and A. P. Prudnikov, *Spravocnik po operacionnomu izcisleniu* (in Russian), p. 265. Vysch. skola, Moskva (1965).
28. Y. L. Luke, *Integrals of Bessel Functions*, p. 122. McGraw-Hill, New York (1962).
29. R. A. Guyer and J. A. Krumhansl, Dispersion relation for second sound in solids, *Phys. Rev.* **133A,** 1411 (1964).
30. B. R. Appleton and G. R. Cellar (Editors), *Laser and Electron-beam Interactions with Solids*. Elsevier Science, New York (1982).

PROPAGATION "NON FOURIERISTE" DES PULSATIONS DE CHALEUR DANS UN MILIEU FINI

Résumé—On étudie analytiquement les effets dans une couche finie de la conduction hyperbolique de la chaleur. Les résultats pour des conditions de flux thermique pulsé à la surface sont comparés avec ceux obtenus à partir de l'équation parabolique classique. Une analyse détaillée de la transition entre comportement "parabolique" et "hyperbolique" de la réponse en température à la pulsation montre que pour $L/\sqrt{(a\tau)} \gtrsim 25$ les effets de la conduction non classique sont négligeables. Les solutions données ont une interprétation physique claire, comme les ondes thermiques, ce qui rend possible l'écriture immédiate des solutions pour un milieu semi-infini. Ces solutions en développement en série présentées ici conviennent pour la convergence numérique et elles permettent de faire une analyse fouillée des premiers instants du phénomène dans le milieu, ce qui est très important dans l'étude des contraintes thermiques.

AUSBREITUNG VON HEIZ-PULSEN IN EINEM ENDLICHEN MEDIUM: NICHT-FOURIER-EFFEKTE

Zusammenfassung—In dieser Arbeit werden mit Hilfe eines hyperbolischen Wärmeleitungs-Modells auf analytische Weise Nicht-Fourier-Effekte untersucht. Die Ergebnisse für pulsierende Wärmestromdichte an der Oberfläche werden mit denjenigen verglichen, welche mit Hilfe der üblichen parabolischen Standard-Wärmeleitgleichung ermittelt wurden. Eine detaillierte Untersuchung des Übergangs zwischen "parabolischem" und "hyperbolischem" Verhalten der Temperatur-Antwort auf die pulsierende Wand-Wärmestromdichte zeigt, daß für $L/\sqrt{(a\tau)} \gtrsim 25$ Nicht-Fourier-Effekte vernachlässigbar sind und die Temperatur an beliebiger Stelle praktisch gleich derjenigen bei "parabolischer" Rechnung ist. Die angegebenen Lösungen lassen sich klar als wandernde thermische Wellen interpretieren, was es ermöglicht, die Lösungen für dieses Problem als halb unendlichen Körper anzugeben. Die gezeigten Reihenentwicklungen haben eine für numerische Konvergenz angenehme Form und ermöglichen eine tiefgreifende Untersuchung der Anfangsphase transienter Änderungen, was im Hinblick auf Wärmespannungen sehr wichtig ist.

РАСПРОСТРАНЕНИЕ ТЕПЛОВЫХ ИМПУЛЬСОВ В ОГРАНИЧЕННОЙ СРЕДЕ, НЕ ПОДЧИНЯЮЩЕЕСЯ ЗАКОНУ ФУРЬЕ

Аннотация—С помощью гиперболической модели теплопроводности в ограниченной пластине аналитически исследуются эффекты, не подчиняющиеся закону Фурье. Результаты для пульсирующего теплового потока на поверхности сравниваются с данными, полученными из стандартного параболического уравнения теплопроводности. Детальный анализ показывает, что для случая $L/\sqrt{(a\tau)} \gtrsim 25$ отклонения от закона Фурье, пренебрежимо малы, а температурный отклик в точке $x = L$ на импульс, возникающий при $x = 0$, практически равен полученному из параболического уравнения. Данные решения имеют ясную физическую интерпретацию в виде движущихся тепловых волн, дающую возможность записать решение этой задачи для полуограниченной среды. Решения, представленные в виде рядов, удобны для численного счета и позволяют провести петальный анализ начальной стадии переходного периода в среде, которая играет важную роль при исследовании термических напряжений.